

# On the Nash Stability in the Hedonic Coalition Formation Games

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## Abstract

This paper studies *the Nash stability* in hedonic coalition formation games. We address the following issue: for a general problem formulation, is there any utility allocation method ensuring a Nash-stable partition? We propose the definition of *the Nash-stable core* and we analyze the conditions for having a non-empty Nash-stable core. More precisely, we prove that using relaxed efficiency in utility sharing allows to ensure a non empty Nash-stable core. Then, a decentralized algorithm called *Nash stability establisher* is proposed for finding the Nash stability in a game whenever at least one exists. The problem of finding the Nash stability is formulated as a non-cooperative game. In the proposed approach, during each round, each player determines its strategy in its turn according to a *random round-robin scheduler*. We prove that the algorithm converges to an equilibrium if it exists, which is the indicator of the Nash stability.

## I. INTRODUCTION

Any cooperation among agents (players) being able to make strategic decisions becomes a *coalition formation game* when the players may for various personal reasons wish to belong to a relative *small coalition* rather than the “grand coalition”<sup>1</sup>. Partitioning players represents the crucial question in the game context and a stable partition of the players is referred to as an equilibrium. In [1], the authors propose an abstract approach to coalition formation that focuses

<sup>1</sup>the grand coalition is the set of all agents

on simple merge and split stability rules transforming partitions of a group of players. The results are parametrized by a preference relationship between partitions from the point of view of each player.

On the other hand, a coalition formation game is called to be *hedonic* if

- *the gain of any player depends solely on the members of the coalition to which the player belongs, and*
- *the coalitions form as a result of the preferences of the players over their possible coalitions' set.*

Accordingly, the stability concepts aiming *hedonic conditions* can be summarized as following [2]: a partition could be *individually stable*, *Nash stable*, *core stable*, *strict core stable*, *Pareto optimal*, *strong Nash stable*, *strict strong Nash stable*. In the sequel, we concentrate on the Nash stability. The definition of the Nash stability is quite simple:

*a partition of players is Nash stable whenever there is no player deviating from his/her coalition to another coalition in the partition.*

We refer to [2] for further discussions concerning the stability concepts in the context of hedonic coalition formation games.

#### A. Relations to Set-partitioning Optimization Problem

#### B. Related Work

In [3], the problem of generating Nash stable solutions in coalitional games is considered. In particular, the authors proposed an algorithm for constructing the set of all Nash stable coalition structures from players' preferences in a given additively separable hedonic game. In [10], a bargaining procedure of coalition formation in the class of hedonic games, where players' preferences depend solely on the coalition they belong to is studied. The authors provide an example of nonexistence of a pure strategy stationary perfect equilibrium, and a necessary and sufficient condition for existence. They show that when the game is totally stable (the game and all its restrictions have a nonempty core), there always exists a no-delay equilibrium generating core outcomes. Other equilibria exhibiting delay or resulting in unstable outcomes can also exist. If the core of the hedonic game and its restrictions always consist of a single point, it is shown that the bargaining game admits a unique stationary perfect equilibrium, resulting in the immediate formation of the core coalition structure.

In [4], Drèze and Greenberg introduced the hedonic aspect in players' preferences in a context concerning local public goods. Moreover, purely hedonic games and stability of hedonic coalition partitions were studied by Bogomolnaia and Jackson in [5]. In this paper, it is proved that if players' preferences are additively separable and symmetric, then a Nash stable coalition partition exists. For further discussion on additively separable and symmetric preferences, we refer the reader to [6].

### C. Contributions

Our work aims at considering stable strategies for hedonic coalition formation games. We first develop a simple decentralized algorithm finding the Nash stability in a game if at least one exists. The algorithm is based on *the best reply strategy* in which each player decides serially his/her coalition. Thus, the problem is considered as a non-cooperative game. We consider a *random round-robin* fashion where each player determines its strategy in its turn according to a *scheduler* which is randomly generated for each round. Under this condition, we prove that the algorithm converges to an equilibrium if it exists.

Then the fundamental question that comes is to determine which utility allocation methods may ensure a Nash-stable partition. We address this issue in the sequel. We first propose the definition of *the Nash-stable core* which is the set of all possible utility allocation methods resulting in Nash-stable partitions. We show that efficient utility allocations where the utility of a group is completely shared between his/her members, may have no Nash-stable partitions with some exceptions. Rather, we proved that relaxing the efficiency condition may ensure the non-emptiness of the core. Indeed we prove that if the sum of players' gains within a coalition is allowed to be less than the utility of this coalition then a Nash-stable partition always exist.

## II. HEDONIC COALITION FORMATION

### A. Definition

A coalition formation game is given by a pair  $\langle N, \succ \rangle$ , where  $N$  is the set of  $n$  players and  $\succ = (\succeq_1, \succeq_2, \dots, \succeq_n)$  denotes the *preference profile*, specifying for each player  $i \in N$  his preference relation  $\succeq_i$ , i.e. a reflexive, complete and transitive binary relation.

*Definition 2.1:* A coalition structure or a *coalition partition* is a set  $\Pi = \{S_1, \dots, S_l\}$  which partitions the players set  $N$ , i.e.,  $\forall k, S_k \subset N$  are disjoint coalitions such that  $\bigcup_{k=1}^l S_k = N$ . Given  $\Pi$  and  $i$ , let  $S_\Pi(i)$  denote the set  $S_k \in \Pi$  such that  $i \in S_k$ .

In its partition form, a coalition formation game is defined on the set  $N$  by associating a utility value  $u(S|\Pi)$  to each subset of any partition  $\Pi$  of  $N$ . In its characteristic form, the utility value of a set is independent of the other coalitions, and therefore,  $u(S|\Pi) = u(S)$ . The games of this form are more restricted but present interesting properties to achieve equilibrium. Practically speaking, this assumption means that the gain of a group is independent of the other players outside the group.

Hedonic coalition formation games fall into this category with an additional assumption:

*Definition 2.2:* A coalition formation game is called to be *hedonic* if

- *the gain of any player depends solely on the members of the coalition to which the player belongs, and*
- *the coalitions form as a result of the preferences of the players over their possible coalitions' set.*

### B. Preference relation

The preference relation of a player can be defined over a *preference function*. Let us denote by  $\pi_i : 2^N \rightarrow \mathbb{R}$  the preference function of player  $i$ . Thus, player  $i$  prefers the coalition  $S$  to  $T$  iff,

$$\pi_i(S) \geq \pi_i(T) \Leftrightarrow S \succeq_i T. \quad (1)$$

We consider the case where the preference relation is chosen to be the utility allocated to the player in a coalition, then  $\pi_i(S) = \phi_i^S$  where  $\phi_i^S$  refers to the utility received by player  $i$  in coalition  $S$ .

In the case of transferable utility games (TU games) we are considering in this paper, the utility of a group can be tranfered among users in any way. thus, an utility allocation is set relatively efficient if for each coalition  $S$ , the sum of individual utilities is equal to the coalition utility:  $\sum_{i \in S} \phi_i^S = u(S), \forall S \subseteq N$ .

Now, if the preferences of a player are *additively separable*, the preference can be even stated with a function characterizing how a player prefers another player in each coalition. This

means that the player's preference for a coalition is based on individual preferences. This can be formalized as follows:

*Definition 2.3:* The preferences of a player are said to be additively separable if there exists a function  $v_i : N \rightarrow \Re$  s.t.  $\forall S, T \subseteq N$

$$\sum_{j \in S} v_i(j) \geq \sum_{j \in T} v_i(j) \Leftrightarrow S \succeq_i T \quad (2)$$

where, according to [5],  $v_i(i)$  is normalized and set to  $v_i(i) = 0$ .

A profile of additively separable preferences, represented by  $(v_i, \dots, v_n)$ , satisfies *symmetry* if  $v_i(j) = v_j(i), \forall i, j$ .

### C. Nash stability

The question we address in this paper concerns the stability of this kind of games. The stability concept for a coalition formation game may receive different definitions. In the literature, a game is either said *individually stable*, *Nash stable*, *core stable*, *strict core stable*, *Pareto optimal*, *strong Nash stable*, *strict strong Nash stable*. We refer to [2] for a thorough definition of these different stability concepts.

In this paper, we concentrate only on the Nash stability because we are interested in those games where the players don't cooperate to take their decisions which means that only individual moves are allowed.

The definition of the Nash stability for an *hedonic coalition formation game* is simply:

*Definition 2.4:* A partition of players is Nash stable whenever no player is incentive to unilaterally change his or her coalition to another coalition in the partition.

which can be mathematically formulated as:

*Definition 2.5 (Nash Stability):* A partition  $\Pi$  is said to be Nash-stable if no player can benefit from moving from his coalition  $S_{\Pi}(i)$  to another existing coalition  $S_k$ , i.e.:

$$\forall i, k : S_{\Pi}(i) \succeq_i S_k \cup \{i\}; S_k \in \Pi \cup \{\emptyset\} \quad (3)$$

Nash-stable partitions are immune to individual movements even when a player who wants to change does not need permission to join or leave an existing coalition [5].

*Remark 2.1:* In the literature ([5], [9]), the stability concepts being immune to individual deviation are *Nash stability*, *individual stability*, *contractual individual stability*. Nash stability

is the strongest within above. We concentrate our analysis on the partitions that are Nash-stable. the notion of *core stability* has been used already in some models where immunity to coalition deviation is required [9]. But the Nash stable core has not been defined yet at the best of our knowledge. This is what we derive in the next section.

Under this definition we propose in this paper to evaluate the existence of Nash stability and to propose an approach that ensures the convergence to a Nash equilibrium of an approximated convex game.

### III. THE NASH-STABLE CORE

#### A. Definiton

Let us consider a hedonic TU game noted  $\langle N, u, \succ \rangle$  (since  $u$  is transferable to the players, we consider hedonic coalition formation games based on transferable utility). For the sake of simplicity, the preference function of player  $i$  is assumed to be the gain obtained in the corresponding coalition, i.e.,  $\pi_i(S) \equiv \phi_i^S, \forall i \in S, \forall S \subseteq N$  as well as let  $\Phi = \{\phi_i^S; \forall i \in S, \forall S \subseteq N\}$  denote the *allocation method* which directly sets up a corresponding preference profile. The corresponding space is equal to the number of set, i.e.  $\Phi \in \mathbb{R}^\kappa$  where  $\kappa = n2^{n-1}$ .

We now define the operator  $\mathbb{F} : \mathbb{R}^\kappa \mapsto \mathcal{P} \cup \emptyset$ , where  $\mathcal{P}$  is the set of all possible partitions. Clearly, for any preference function, the operator  $\mathbb{F}$  finds the set of Nash-stable partition  $\Pi$ , i.e.  $\mathbb{F}(\Phi) := \{\Pi \mid \text{Nash-stable}\}$ . If a Nash-stable partition can not be found, the operator maps to empty set. Moreover, the inverse of the operator is denoted as  $\mathbb{F}^{-1}(\Pi \in \mathcal{P})$  which finds the set of all possible preference functions that give the Nash-stable partition  $\Pi$ . Thus, the Nash-stable core includes all those efficient allocation methods that build the following set:

$$\mathcal{N}\text{-core} = \left\{ \Phi \in \mathbb{R}^\kappa \mid \exists \Pi \in \mathcal{P} ; \mathbb{F}^{-1}(\Pi) \ni \Phi \right\}. \quad (4)$$

#### B. Non-emptiness

To know if the core is non empty, we need to state the set of constraints the partition function as to fulfill. Under the assumption of *efficiency*, we have a first set of constraints

$$\mathcal{C}_{\text{Efficient}}(\Phi) := \left\{ \sum_{j \in S} \phi_j^S = u(S), \forall S \subseteq N \right\}. \quad (5)$$

Then a given partition  $\Pi$  is Nash-stable with respect to a given partition function if the following constraints hold:

$$\mathcal{C}_{\text{Nash-stable}}(\Pi, \Phi) := \left\{ \phi_i^{S_{\Pi}(i)} \geq \phi_i^{T \cup i}, \forall T \in \Pi \cup \emptyset; \forall i \in N \right\}, \quad (6)$$

where  $S_{\Pi}(i)$  is the unique set in  $\Pi$  containing  $i$ . Then, the Nash-stable core is non-empty, iif:

$$\exists \{\Phi^*, \Pi^*\} \mid \mathcal{C}_{\text{Efficient}}(\Phi^*) \text{ and } \mathcal{C}_{\text{Nash-stable}}(\Pi^*, \Phi^*) \quad (7)$$

The Nash-stable core can be further defined as:

$$\mathcal{N}\text{-core} = \left\{ \Phi \in \mathcal{R}^N \mid \exists \Pi \in \mathcal{P}; \mathcal{C}_{\text{Efficient}}(\Phi) \text{ and } \mathcal{C}_{\text{Nash-stable}}(\Pi, \Phi) \right\}, \quad (8)$$

which let us to conclude:

*Theorem 3.1:* The Nash-stable core can be non-empty.

*Proof:* Algorithmically, the Nash-stable core is non-empty if the following linear program is feasible:

$$\min_{\Phi \in \mathcal{R}^N} \left\{ \sum_{\forall S \subseteq N} \sum_{\forall i \in S} \phi_i^S \mid \exists \Pi \in \mathcal{P}; \mathcal{C}_{\text{Efficient}}(\Phi) \text{ and } \mathcal{C}_{\text{Nash-stable}}(\Pi, \Phi) \right\}. \quad (9)$$

■

However, it is not possible to state about the non-emptiness of the Nash-stable core in the general case. Further, searching in an exhaustive manner over the whole partitions is NP-complet as the number of partitions grows according to the Bell number. Typicall, with only 10 players, the number of partition is as large as 115,975.

We now analyse some specific cases in the following.

### C. Grand Coalition Stability Conditions

In the case the grand coalition is targeted, the stability conditions are the following. Let  $\Pi = \{N\}$ , then the following constaints hold:

$$\phi_i^N \geq u(i), \forall i \in N, \quad (10)$$

$$\sum_{i \in N} \phi_i^N = u(N). \quad (11)$$

Resulting in the following:

$$u(N) \geq \sum_{i \in N} u(i). \quad (12)$$

Those cooperative TU games that satisfy this condition are said to be *essential*.

#### D. Symmetric Relative Gain

We now propose to formulate a special game where the utility is shared among players with an equal relative gain.

Let us denote the gain of player  $i$  in coalition  $S$  as  $\phi_i^S = u(i) + \delta_i^S$  in which  $\delta_i^S$  is called *the relative gain*. Note that for an isolated player  $i$ , one have  $\delta_i^i = 0$ . The preference relation can be determined w.r.t. the relative gain:

$$\delta_i^S \geq \delta_i^T \Leftrightarrow S \succeq_i T. \quad (13)$$

The total allocated utilities in coalition  $S$  is  $\sum_{i \in S} \phi_i^S = \sum_{i \in S} u(i) + \sum_{i \in S} \delta_i^S = u(S)$ . Therefore,  $\sum_{i \in S} \delta_i^S = u(S) - \sum_{i \in S} u(i) = \Delta(S)$ , where  $\Delta(S)$  is the *marginal utility* due to coalition  $S$ .

The symmetric relative gain sharing approach relies on equally dividing the marginal utility in a coalition, i.e.

$$\delta_i^S = \frac{\Delta(S)}{|S|}, \quad \forall i \in S. \quad (14)$$

This choice means that each player in coalition  $S$  has the same gain; thus the effect of coalition  $S$  is identical to the players within it.

**Corollary 3.1: Equivalent Evaluation:** Assume that  $S \cap T \neq \emptyset$ . Due to eq. (14), the following must hold

$$\frac{\Delta(S)}{|S|} \geq \frac{\Delta(T)}{|T|} \Leftrightarrow S \succeq_i T \quad \forall i \in S \cap T. \quad (15)$$

It means that all players in  $S \cap T$  prefer coalition  $S$  to  $T$  whenever the relative gain in  $S$  is higher than  $T$ .

For this particular case we obtained the following theorems:

**Lemma 3.1:** There is always a Nash-stable partition when  $N = \{1, 2\}$  in case of symmetric relative gain.

*Proof:* see appendix ■

**Lemma 3.2:** There is always a Nash-stable partition when  $N = \{1, 2, 3\}$  in case of symmetric relative gain.

*Proof:* see appendix ■

Thus, we can conclude that symmetric relative gain always results in a Nash-stable partition when  $n \leq 3$ . However, this is not the case when  $n > 3$ . We can find many counter examples that justify it such as the following one:



*Counter Example 3.1:* Let the marginal utility for all possible  $S$  be as following:

$$\begin{aligned} \Delta(1, 2) = 0.86, \Delta(1, 3) = 0.90, \Delta(1, 4) = 0.87, \Delta(2, 3) = -1.22, \Delta(2, 4) = -1.25, \Delta(3, 4) = -1.21, \\ \Delta(1, 2, 3) = 0.27, \Delta(1, 2, 4) = 0.24, \Delta(1, 3, 4) = 0.28, \Delta(2, 3, 4) = -1.84, \Delta(1, 2, 3, 4) = -0.35 \end{aligned} \quad (16)$$

Let us now generate the preference profile according to these marginal utility values. Notice that we could eliminate those marginal utilities which are negative since a player will prefer to be alone instead of a negative relative gain. Further, ranking the positive relative gains in a descending order results in the following sequence:

$$\left[ \frac{\Delta(1, 3)}{2}, \frac{\Delta(1, 4)}{2}, \frac{\Delta(1, 2)}{2}, \frac{\Delta(1, 3, 4)}{3}, \frac{\Delta(1, 2, 3)}{3}, \frac{\Delta(1, 2, 4)}{3} \right]. \quad (17)$$

According to the ranking sequence, we are able to generate the preference list of each player:

$$\begin{aligned} (1, 3) \succ_1 (1, 4) \succ_1 (1, 2) \succ_1 (1, 3, 4) \succ_1 (1, 2, 3) \succ_1 (1, 2, 4) \succ_1 (1) \\ (1, 2) \succ_2 (1, 2, 3) \succ_2 (1, 2, 4) \succ_1 (2) \\ (1, 3) \succ_3 (1, 3, 4) \succ_3 (1, 2, 3) \succ_3 (3) \\ (1, 4) \succ_4 (1, 3, 4) \succ_4 (1, 2, 4) \succ_4 (4). \end{aligned} \quad (18)$$

Note that this preference profile does not admit any Nash-stable partition. Thus, we conclude that symmetric relative gain allocation doesn't provide always a Nash-stable partition when  $n > 3$ .

#### *E. Additively Separable and Symmetric Utility Case*

We now turn out to the case of separable and symmetric utility case. Consider eq. (2) meaning that player  $i$  gains  $v_i(j)$  from player  $j$  in any coalition. In case of symmetry,  $v_i(j) = v_j(i) = v(i, j)$  such that  $v(i, i) = 0$ . Further, we denote as  $\phi_i^S = u(i) + \sum_{j \in S} v(i, j)$  the utility that player  $i$  gains in coalition  $S$ . Then, the sum of allocated utilities in coalition  $S$  is given by

$$\sum_{i \in S} \phi_i^S = \sum_{i, j \in S} v(i, j) + \sum_{i \in S} u(i) = u(S). \quad (19)$$

Let us point out that  $\sum_{i, j \in S} v(i, j) = 2 \sum_{i, j \in S: j > i} v(i, j)$  (for example,  $S = (1, 2, 3)$ ,  $\sum_{i, j \in S} v(i, j) = 2[v(1, 2) + v(1, 3) + v(2, 3)]$ ). Therefore, the following determines the existence of additively separable and symmetric preferences when the utility function  $u$  is allocated to the players:

$$\sum_{i, j \in S: j > i} v(i, j) = \frac{1}{2} \left( u(S) - \sum_{i \in S} u(i) \right), \quad (20)$$

where  $\Delta(S) = \frac{1}{2} (u(S) - \sum_{i \in S} u(i))$  is the *marginal utility* due to coalition  $S$ .

Albeit we will come back later on this point, it is true noting these constraints are strongly restrictive and are rarely observed in real problems. Let it be illustrated for example  $N = (1, 2, 3)$

The constraints imposed are:

$$\begin{aligned} v(1, 2) &= \frac{1}{2}[u(1, 2) - u(1) - u(2)], \\ v(1, 3) &= \frac{1}{2}[u(1, 3) - u(1) - u(3)], \\ v(2, 3) &= \frac{1}{2}[u(2, 3) - u(2) - u(3)], \\ v(1, 2) + v(1, 3) + v(2, 3) &= \frac{1}{2}[u(1, 2, 3) - u(1) - u(2) - u(3)]. \end{aligned} \quad (21)$$

We have 3 variables and 4 constraints, thus only special problem may fit with all constraints. A more general approach allows to state the following theorem:

*Theorem 3.2:* The Nash-stable core of the Additively Separable and Symmetric Utility Hedonic game is non-empty if there exist balanced weights of the dual problem (balancedness conditions):

$$\exists \{w_S, \forall S \in 2^N\} : \quad (22)$$

$$\begin{aligned} \sum_{S \in 2^N} w_S \Delta(S) &\leq \Delta(N), \\ \sum_{S \in 2^N: \mathcal{V}(k) \in S} w_S &= 1, \quad \forall k \in \mathcal{I}, \\ w_S &\in [-\infty, \infty] \quad \forall S \in 2^N. \end{aligned} \quad (23)$$

*Proof:* According to Bondareva-Shapley theorem [11], [12] when the gains of players are allocated according to the additively separable and symmetric way we can note:

- $\mathcal{V}$  the all possible bipartite coalitions such that  $\mathcal{V} := \{(i, j) \in 2^N : j > i\}$ . Note that  $|\mathcal{V}| = n(n-1)/2$ .
- $\mathcal{I}$  the index set of all possible bipartite coalitions. So,  $\mathcal{V}(k \in \mathcal{I})$  is the  $k$ th bipartite coalition.
- $\mathbf{v} = (v(i, j))_{(i, j) \in \mathcal{V}} \in \mathbb{R}^{|\mathcal{V}|}$  which is the vector demonstration of all  $v(i, j)$ .
- $\mathbf{c} = (1, \dots, 1) \in \mathbb{R}^{|\mathcal{V}|}$ .
- $\mathbf{b} = (b_S)_{S \in 2^N} \in \mathbb{R}^{2^n}$  such that  $b_S = \frac{1}{2}\Delta(S)$  where  $\Delta(S) = u(S) - \sum_{i \in S} u(i)$ .
- $\mathbf{A} = (a_{S,k})_{S \in 2^N, k \in \mathcal{I}} \in \mathbb{R}^{2^n \times |\mathcal{V}|}$  is a matrix such that  $a_{S,k} = \mathbf{1}\{S : \mathcal{V}(k) \in S\}$ .

By using these definitions, the Nash-stable core is non-empty whenever the following linear program is feasible

$$(L) \quad \min \mathbf{c}\mathbf{v} \quad \text{subject to} \\ \mathbf{A}\mathbf{v} = \mathbf{b}, \mathbf{b} \in [-\infty, \infty]. \quad (24)$$

The linear program that is dual to (L) is given by

$$(\hat{L}) \quad \max \mathbf{w}\mathbf{b} \quad \text{subject to} \\ \mathbf{w}\mathbf{A} = \mathbf{c}, \mathbf{w} \in [-\infty, \infty], \quad (25)$$

where  $\mathbf{w} = (w_S)_{S \in 2^N} \in \mathbb{R}^{2^N}$  denote the vector of dual variables. Let  $\mathbf{A}^k$  denote the  $k$ th column of  $\mathbf{A}$ . Then  $\mathbf{w}\mathbf{A}$  implies

$$\mathbf{w}\mathbf{A}^k = \sum_{S \in 2^N} w_S a_{S,k} = \sum_{S \in 2^N: \mathcal{V}(k) \in S} w_S = 1, \quad \forall k \in \mathcal{I}. \quad (26)$$

This result means that the feasible solutions of  $(\hat{L})$  exactly correspond to the vectors containing balancing weights for balanced families. More precisely, when  $\mathbf{w}$  is feasible in  $(\hat{L})$ ,  $\mathcal{B}_{\mathbf{w}}$  is a balanced family with *balancing weights*  $(w_S)_{S \in \mathcal{B}_{\mathbf{w}}}$ .

According to the *weak duality theorem*, the objective function value of the primal (L) at any feasible solution is always greater than or equal to the objective function value of the dual  $(\hat{L})$  at any feasible solution, i.e.  $\mathbf{c}\mathbf{v} \geq \mathbf{w}\mathbf{b}$  which implies

$$\mathbf{c}\mathbf{v} = \sum_{k \in \mathcal{I}} v_k = \sum_{\forall S \in 2^N} \sum_{i,j \in S: j > i} v(i,j) = \frac{1}{2} \Delta(N), \quad (27)$$

and

$$\mathbf{w}\mathbf{b} = \frac{1}{2} \sum_{S \in 2^N} w_S \Delta(S) \leq \frac{1}{2} \Delta(N). \quad (28)$$

Combining these results, we have the following balancedness conditions of  $u$ :

$$\begin{aligned} \sum_{S \in 2^N} w_S \Delta(S) &\leq \Delta(N), \\ \sum_{S \in 2^N: \mathcal{V}(k) \in S} w_S &= 1, \quad \forall k \in \mathcal{I}, \\ w_S &\in [-\infty, \infty] \quad \forall S \in 2^N. \end{aligned} \quad (29)$$

■

However, these conditions are very restrictive. For a given set of players  $|N| = n$ , the number of variables is strictly equal to  $\binom{2}{n} = n \cdot (n-1)/2$  while the number of constraints is equal to the number of sets, i.e.  $2^n - 1$ . For  $n = 10$ , we have 45 variables and as much as 512 constraints.

#### F. Relaxed Efficiency

Considering the former result we propose to transform an initial Hedonic game under an Additively Separable and Symmetric Utility Case, by relaxing the efficiency constraint. Clearly, we propose to relax the constraint of having the sum of allocated utilities in a coalition to be strictly equal to the utility of the coalition, i.e.  $\sum_{i \in S} \phi_i^S \leq u(S)$ .

We assume that the system cannot provide any coalition with additional utility and therefore the unique way is to tax a group to ensure the convergence, which leads to having:

$$\sum_{i,j \in S: j > i} v(i, j) \leq \frac{1}{2} \Delta(S). \quad (30)$$

Now the following theorem may be stated:

*Theorem 3.3:* The Nash-stable core is always non-empty in case of relaxed efficiency.

*Proof:* A feasible solution of the following linear program guarantees the non-emptiness of the Nash-stable core:

$$\begin{aligned} \max \quad & \sum_{\forall S \in 2^N} \sum_{i,j \in S: j > i} v(i, j) \text{ subject to} \\ & \sum_{i,j \in S: j > i} v(i, j) \leq \frac{1}{2} \Delta(S), \forall S \in 2^N, \end{aligned} \quad (31)$$

which is equivalent to

$$\begin{aligned} (LRE) \quad & \max \mathbf{c} \mathbf{v} \text{ subject to} \\ & \mathbf{A} \mathbf{v} \leq \mathbf{b}, \mathbf{b} \in [-\infty, \infty]. \end{aligned} \quad (32)$$

Note that  $(LRE)$  is always feasible since

- there are no any inconsistent constraints, i.e. there are no at least two rows in  $\mathbf{A}$  that are equivalent,
- the polytope is bounded in the direction of the gradient of the objective function  $\mathbf{c} \mathbf{v}$ .

■

### G. Linear Least Square

In the case where the system is able to provide some redistribution of utilities to a coalition, we can also allow the sum of individual utilities in a coalition to be higher than the coalition's utility. In this case, the system may be interested however to find an Additively Separable and Symmetric Utility while minimizing the total deviation from the utilities. We can then propose to select the symmetric preferable preferences  $v(i, j)$  according to the following optimization problem:

$$(LLS) \quad \min \|\mathbf{b} - \mathbf{A}\mathbf{v}\|. \quad (33)$$

This formulation leads to an analytical solution:  $\mathbf{v} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}$ . However the system may be interested in adding hard constraints on some specific sets, typically the grand coalition, to avoid the risk of having to pay some additional costs. This can be done by adding additional constraints to the problem above, but if the constraints are expressed as linear equalities or inequalities the problem remains convex.

## IV. HEDONIC COALITION FORMATION AS A NON-COOPERATIVE GAME

In this section, we now develop a decentralized algorithm to reach a Nash-stable partition whenever one exists in a hedonic coalition formation game.

We in fact model the problem of finding a Nash stable partition in a hedonic coalition formation by formulating it as a non-cooperative game and we state the following:

*Theorem 4.1:* A hedonic coalition formation game is equivalent to a Non-cooperative game.

*Proof:* Let us denote as  $\Sigma$  the set of strategies. We assume that the number of strategies is equal to the number of players. This is indeed sufficient to represent all possible choices. Indeed, the players that select the same strategy are interpreted as a coalition. ■

Based on this equivalence, it is possible to reuse some classical results from game theory. We consider a *random round-robin* algorithm where each player determines its strategy in its turn according to a *scheduler* which is randomly generated for each round. A scheduler in round  $\ell$  is denoted as  $\mathbf{t}(\ell) = \{t_1(\ell), t_2(\ell), \dots, t_n(\ell)\}$  where  $t_i(\ell)$  is the turn of player  $i$ . It should be noted that a scheduler is a random permutation of the set of players  $N$  and therefore, our problem turns out to a *weakly acyclic games*. A non-cooperative game is classified as weakly acyclic if every strategy-tuple is connected to some pure-strategy Nash equilibrium by a best-reply path. Weakly

acyclic games have the property that if the order of deviators is decided more-or-less randomly, and if players do not deviate simultaneously, then a best-reply path reaches an equilibrium [8] if there exists at least one.

*Remark 4.1:* The type of scheduler chosen in this work is by default considered as *memoryless* since the identity of the deviator in each round does not depend on the previous rounds. However, it may be more efficient to design a scheduler according to the past observations. Thus, we come up with so called *algorithmic mechanism design* that could enable to converge an equilibrium in less number of rounds. This kind of optimization is kept out of the scope of this paper.

A *strategy tuple* in step  $s$  is denoted as  $\sigma^{(s)} = \{\sigma_1^{(s)}, \sigma_2^{(s)}, \dots, \sigma_n^{(s)}\}$ , where  $\sigma_i^{(s)}$  is the strategy of player  $i$  in step  $s$ . The relation between a round and a step can be given by  $\ell = \lceil \frac{s}{n} \rceil$ . In each step, only one dimension is changed in  $\sigma^{(s)}$ . We further denote as  $\Pi^{(s)}$  the partition in step  $s$ . Define as  $S_i^{(s)} = \{j : \sigma_i^{(s)} = \sigma_j^{(s)}, \forall j \in N\}$  the set of players that share the same strategy with player  $i$ . Thus, note that  $\cup_{i \in N} S_i^{(s)} = N$  for each step. The preference function of player  $i$  is denoted as  $\pi_i(\sigma^{(s)})$  and verifies the following equivalence:

$$\pi_i(\sigma^{(s)}) \geq \pi_i(\sigma^{(s-1)}) \Leftrightarrow S_i^{(s)} \succeq_i S_i^{(s-1)}, \quad (34)$$

where player  $i$  is the one that takes its turn in step  $s$ .

Any sequence of strategy-tuple in which each strategy-tuple differs from the preceding one in only one coordinate is called a *path*, and a unique deviator in each step strictly increases the utility he receives is an *improvement path*. Obviously, any *maximal improvement path* which is an improvement path that can not be extended is terminated by stability.

An algorithm for hedonic coalition formation can be given as following:

*Theorem 4.2:* The proposed Algorithm 1 (Nash stability establisher) always converges to a stable partition whenever there exists one.

*Proof:* The proof exploits the property mentioned above, relative to weakly acyclic games: *every weakly acyclic game admits always a Nash equilibrium [7]; since Nash stability establisher is exactly a weakly acyclic game, it always converges to a partition which is Nash-stable.*

Let us denote as  $\Pi^{(0)}$  the initial partition where each player is alone. It corresponds to the case where each player chooses different strategy; thus each player is alone in its strategy:  $S_i^{(0)} = i, \forall i \in N$ . The transformation of strategy tuple and partition in each step can be denoted

**Algorithm 1** Nash stability establisher

---

```

set stability flag to zero
while stability flag is zero do
    generate a scheduler
    according to the scheduler, each player chooses the best-reply strategy
    check stability
    if stability found then
        set stability flag to one
    end if
end while

```

---

as following:

$$\sigma^{(0)} \rightarrow \sigma^{(1)} \rightarrow \dots \rightarrow \sigma^{(s^*)} \Rightarrow \Pi^{(0)} \rightarrow \Pi^{(1)} \rightarrow \dots \rightarrow \Pi^{(s^*)}, \quad (35)$$

where  $s^*$  represents the the step in which the stable partition occurs. In fact, the stable partition in  $\Pi^{(s^*)}$  is exactly the Nash equilibrium in a weakly acyclic game. ■

## V. EVALUATION AND ILLUSTRATIVE EXAMPLE

### A. Evaluation Framework and Objective Functions

We propose now to evaluate the use of the former algorithm on hedonic games transformed to additively separable and symmetric.

1) *Partly centralized approach*: Such approach needs two steps. During the first step, the system computes the relative symmetric gains  $v(i, j)$  according to one of the suboptimal approximations proposed above. Then, during the second step, the players do their moves according to the algorithm and using the modified utilities until an equilibrium is reached.

2) *Social optimum*: The social optimum is the maximum total global utility, i.e. what is the partitioning of the players such that the total global utility is maximized. It can be formulated as a set partitioning optimization problem which can be given by

$$S_u^* = \max_{\Pi} \sum_{S \in \Pi} u(S), \quad (36)$$

by which we find a partition  $\Pi^*$  maximizing the global utility. Note that the total social utility in case of a Nash-stable partition will always be lower or equal to the one obtained by social optimum, i.e.  $S_u \leq S_u^*$ . We could say that an approach is socially optimal if it reaches the social optimum. The distance between the social optimum and the equilibrium solution achieved is the price of anarchy.

### B. A small size example

We study here the utility allocation based on relaxed efficiency according to the marginal utilities given in Counter Example 3.1. We also suppose that  $u(1) = 0.15$ ,  $u(2) = 1.68$ ,  $u(3) = 0.01$ ,  $u(4) = 1.78$ . We calculate the social optimum which is equivalent to the set partitioning problem's optimization version.

1) *Linear programming approach:* A utility allocation method can be found by solving the following linear program:

$$\begin{aligned}
 & \max\{v(1, 2) + v(1, 3) + v(1, 4) + v(2, 3) + v(2, 4) + v(3, 4)\} \text{ subject to} \\
 & v(1, 2) \leq \frac{0.86}{2}, v(1, 3) \leq \frac{0.90}{2}, v(1, 4) \leq \frac{0.87}{2}, \\
 & v(2, 3) \leq -\frac{1.22}{2}, v(2, 4) \leq -\frac{1.25}{2}, v(3, 4) \leq -\frac{1.21}{2}, \\
 & v(1, 2) + v(1, 3) + v(2, 3) \leq \frac{0.27}{2}, v(1, 2) + v(1, 4) + v(2, 4) \leq \frac{0.24}{2}, \\
 & v(1, 3) + v(1, 4) + v(3, 4) \leq \frac{0.28}{2}, v(2, 3) + v(2, 4) + v(3, 4) \leq -\frac{1.84}{2}, \\
 & v(1, 2) + v(1, 3) + v(1, 4) + v(2, 3) + v(2, 4) + v(3, 4) \leq -\frac{0.35}{2}, \tag{37}
 \end{aligned}$$

which produces the following values:

$$\begin{aligned}
 & v(1, 2) = 0.3725, v(1, 3) = 0.3724, v(1, 4) = 0.3723, \\
 & v(2, 3) = -0.6100, v(2, 4) = -0.6250, v(3, 4) = -0.6050. \tag{38}
 \end{aligned}$$

Thus, the preference profile is obtained

$$\begin{aligned}
 & (1, 2, 3, 4) \succ_1 (1, 2, 3) \succ_1 (1, 2, 4) \succ_1 (1, 4) \succ_1 (1, 3, 4) \succ_1 (1, 2) \succ_1 (1) \\
 & (1, 2) \succ_2 (2) \\
 & (1, 3) \succ_3 (3) \\
 & (1, 4) \succ_4 (4), \tag{39}
 \end{aligned}$$



which admits the Nash-stable partition  $\Pi = \{(1, 4), (2), (3)\}$ . The total social utility can be calculated as  $S_u = u(1, 4) + u(2) + u(3) = \Delta(1, 4) + u(1) + u(4) + u(2) + u(3) = 4.49$ .

The social optimum in the considered example is found to be  $S_u^* = 4.52$  which is a result of partition  $\Pi^* = \{(1, 3), (2), (4)\}$ .

## VI. CONCLUSIONS

We suggested a decentralized algorithm for finding the Nash stability in a game whenever there exists always at least one. The problem of finding the Nash stability is considered as a non-cooperative game. We consider a *random round-robin* fashion where each player determines its strategy in its turn according to a *scheduler* which is randomly generated for each round. Under this condition, we proved that the algorithm converges to an equilibrium which is the indicator of the Nash stability. Moreover, we answer the following question: Is there any utility allocation method which could result in a Nash-stable partition? We proposed the definition of the Nash-stable core. We analyzed the cases in which the Nash-stable core is non-empty, and prove that in case of the relaxed efficiency condition there exists always a Nash-stable partition.

## VII. APPENDICES

### A. Appendix 1: proof of Lemma 3.1

The proof is based on a simple enumeration of all possible partitions and corresponding conditions of Nash-stability:

1)  $\Pi = \{(1), (2)\}$ :

$$\begin{aligned} 0 &\geq \delta_1^{12} \\ 0 &\geq \delta_2^{12} \\ \delta_1^{12} + \delta_2^{12} &= \Delta(1, 2) \end{aligned} \tag{40}$$

2)  $\Pi = \{1, 2\}$ :

$$\begin{aligned} 0 &\leq \delta_1^{12} \\ 0 &\leq \delta_2^{12} \\ \delta_1^{12} + \delta_2^{12} &= \Delta(1, 2) \end{aligned} \tag{41}$$

According to Corollary 3.1,  $\delta_1^{12} = \delta_2^{12} = \delta = \frac{\Delta(1,2)}{2}$ . Thus, combining all constraint sets of all possible partitions, we have the following result constraint set:  $\mathcal{C}_\Pi := \{0 \leq \delta\} \cup \{0 \geq \delta\} \Leftrightarrow \delta \in [-\infty, \infty]$ . It means that for any value of  $\Delta(1, 2)$ , symmetric relative gain always results in a Nash-stable partition for two players case.

*B. Appendix 2: proof of Lemma 3.2*

Note that there are 5 possible partitions in case of  $N = \{1, 2, 3\}$ . Thus, according to equally divided marginal utility, the following variables occur:  $\delta_1^{12} = \delta_2^{12} = \delta_1$ ,  $\delta_1^{13} = \delta_3^{13} = \delta_2$ ,  $\delta_2^{23} = \delta_3^{23} = \delta_3$ ,  $\delta_1^{123} = \delta_2^{123} = \delta_3^{123} = \delta_4$ . Enumerating all possible partitions results in the following conditions:

1)  $\Pi = \{(1), (2), (3)\}$ :

$$\delta_1 \leq 0, \delta_2 \leq 0, \delta_3 \leq 0, \quad (42)$$

2)  $\Pi = \{(1, 2), (3)\}$ :

$$\delta_1 \geq 0, \delta_1 \geq \delta_2, \delta_1 \geq \delta_3, \delta_4 \leq 0, \quad (43)$$

3)  $\Pi = \{(1, 3), (2)\}$ :

$$\delta_2 \geq 0, \delta_2 \geq \delta_1, \delta_2 \geq \delta_3, \delta_4 \leq 0, \quad (44)$$

4)  $\Pi = \{(2, 3), (1)\}$ :

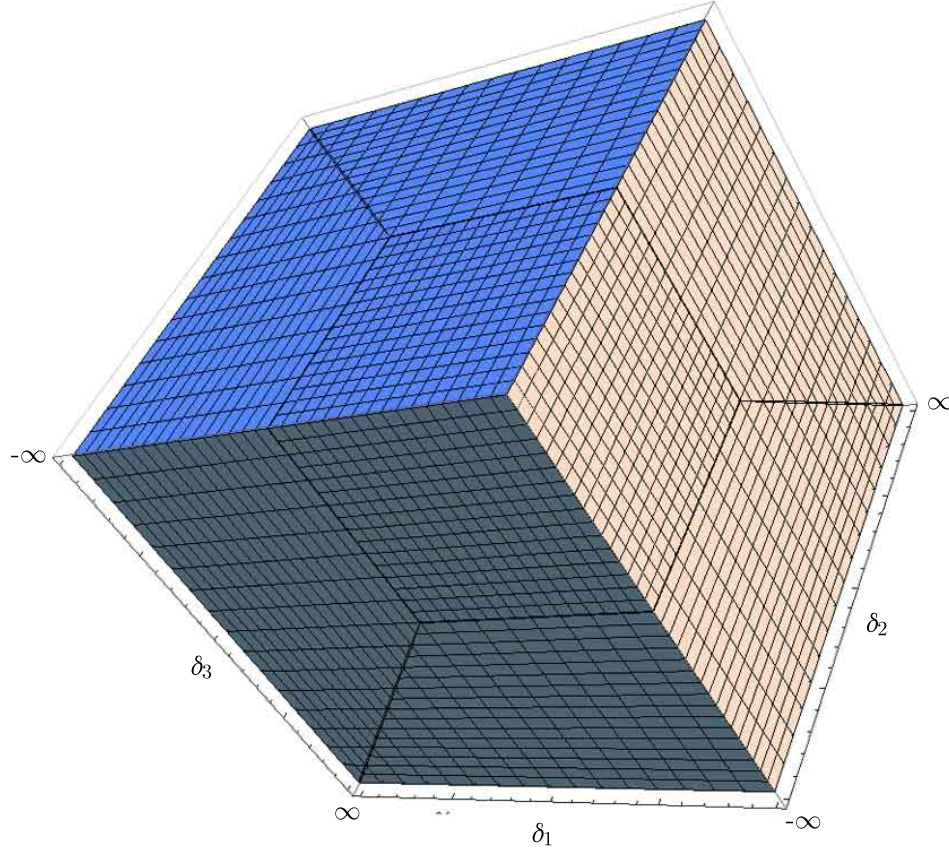
$$\delta_3 \geq 0, \delta_3 \geq \delta_1, \delta_3 \geq \delta_2, \delta_4 \leq 0, \quad (45)$$

5)  $\Pi = \{1, 2, 3\}$ :

$$\delta_4 \geq 0. \quad (46)$$

Note that the constraint set  $\mathcal{C}_\Pi$  covers all values in  $\delta_1, \delta_2, \delta_3$  in case of  $\delta_4 \geq 0$ . Further, it also covers all values when  $\delta_4 \leq 0$ . We are able to draw it since there are three dimensions:

$$\delta_4 \leq 0$$



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